Some non-uniqueness results on stationary distributions for McKean-Vlasov SDEs

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Outline

♣ McKean-Vlasov SDEs and phase transitions

♣ Existence of stationary distribution for DDSDEs

♣ Non-uniqueness

♣ S.-Q. Zhang, Existence and non-uniqueness of stationary distributions for distribution dependent SDEs, Electron. J. Probab. 28 (2023), article no. 93, 1–34.

♣ Some recent results

McKean-Vlasov SDEs

♣ The empirical measure of the position of N particles $\{X_t^i\}_{i=1}^N$

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i},$$

the particles $\{X_t^i\}_{i=1}^N$ satisfy the interaction diffusions:

$$\mathrm{d}X_t^{i,N} = \frac{1}{N}\sum_{j=1}^N b(X_t^{i,N}, X_t^{j,N})\mathrm{d}t + \sigma \mathrm{d}W_t^i,\ \ i = 1, \cdots, N,$$

where $\{W_t^i\}_i^N$ are independent Brownian motions on $\mathbb{R}^d$.

♣ The convergence of μ_t^N as $N \to +\infty$ in weak topology of $\mathscr{P}(\mathbb{R}^d)$ (probability measures on $\mathbb{R}^d$), is called "propagation of chaos" which was introduced by Kac [1] inspired by the work of Boltzmann.

[1] ♣ M. Kac, Foundations of kinetic theory, in Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954-1955, Vol. III, University of California Press, Berkeley and Los Angeles, 1956, pp. 171-197.

♣ McKean [2] prove that for $b(x,y)\equiv F(x-y)$ for some Lipschitz function F, μ_t^N convergence in law to some probability measure μ_t on $\mathbb{R}^d$, and $\mu_t(\mathrm{d}x)=\rho_t(x)\mathrm{d}x$ satisfies (called McKean-Vlasov equatioin)

$$\partial_t\rho_t=\frac{\sigma^2}{2}\Delta\rho_t-\mathrm{div}\left(\rho_t\nabla\left(\int_{\mathbb{R}^d}F(\cdot-y)\mu_t(\mathrm{d}y)\right)\right),\quad t>0.$$

♣ A $\mathbb{R}^d$-value nonlinear process on a filtered probability space $(\Omega,\mathscr{F}_0,\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$ with a B.M. $\{W_t\}_{t\geq 0}$ was also introduced:

$$X_t=X_0+\int_0^t\int_{\mathbb{R}^d}b(X_s,y)\mathscr{L}_{X_s}(\mathrm{d}y)\mathrm{d}s+\sigma W_t,\quad t\geq 0$$

X_0 is some $\mathbb{R}^d$ r.v. and $\mathscr{L}_{X_s}$ is the law of X_s under $\mathbb{P}$. Moreover, $\mathscr{L}_{X_t}$ satisfies the parabolic PDE of μ_t.

[2] ♣ H. P. McKean, Jr., A class of Markov processes associated with nonlinear parabolic equations, Proc. Natl. Acad. Sci. USA, 56 (1966), pp. 1907-1911.

♣ H. P. McKean, Jr., Propagation of Chaos for a Class of Nonlinear Parabolic Equations, in Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic University, 1967), Air Force Office Sci. Res., Arlington, VA, 1967, pp. 41-57.

Phase transitions

♣ For $d = 1$ and $b(x, y) = -x^3 + x - \alpha(x - y)$, Dawson proved that this system has phase transitions [3]: there is $\sigma_c > 0$ such that

for $0 < \sigma < \sigma_c$, there exist three stationary distributions;

for $\sigma \geq \sigma_c$, there exists only one stationary distribution.

♣ The following PDE has three solution if $\sigma \in (0, \sigma_c)$:

$$0 = \frac{\sigma^2}{2}\Delta\rho(x) - \mathrm{div}\left(\rho(x)\left(-x^3 + x - \alpha\int_{\mathbb{R}^d}(x-y)\rho(y)\mathrm{d}y\right)\right),$$

$$\rho \geq 0, \quad \int_{\mathbb{R}^d}\rho(x)\mathrm{d}x = 1.$$

0 is not a simple eigenvalue.

♣ $\frac{1}{\sqrt{2}} \leq \frac{\sigma_c}{\alpha} \leq \sqrt{2}$. Fix σ. Then the phase transition occurs when α is large and does not occur when α is small.

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[3] ♣ D. A. Dawson, Critical dynamics and fluctuations for a mean-field model of cooperative behavior. *J. Stat. Phys.* **31(1)**, 29–85, 1983.

♣ Set $b(x,y) = -\nabla V(x) - \nabla F(x-y)$,

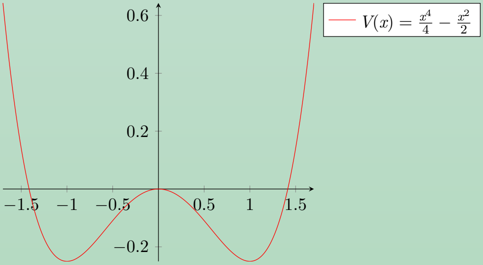
V : the confining potential $\quad F$: the interaction potential

Stationary distributions are of an explicit formulation:

$$\mu(\mathrm{d}x) = \frac{\exp\left\{-\frac{2}{\sigma^2}\left(V(x) + F * \mu(x)\right)\right\}}{\int_{\mathbb{R}^d} \exp\left\{-\frac{2}{\sigma^2}\left(V(x) + F * \mu(x)\right)\right\} \mathrm{d}x} \mathrm{d}x.$$

Dawson's $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$: double-well landscape

$F(x) = \frac{\alpha}{2} x^2$



♣ For polynomial potentials V ($\deg V \geq 4$) and F (even polynomial funciton), existence of several stationary distributions investigated by combining the explicit formulation with free energy functional:

$$\mathscr{E}^{V,F}(\mu) := \frac{\sigma^2}{2} Ent(\mu|\mu_{V_\sigma}) + \frac{1}{2}\mu(F * \mu)$$

where $\mu_{V_\sigma}(\mathrm{d}x) = Z_0 \exp\{-2\sigma^{-2} V(x)\}\mathrm{d}x$ (a probability measure) $\mathrm{Ent}(\mu|\mu_{V_\sigma})$ is the classical entropy. [4]

[4] ♣ J. Tugaut, Convergence to the equilibria for self-stabilizing processes in double well landscape, *Ann. Probab.*, 41 (3) (2013), pp. 1427-1460

♣ J. Tugaut, Phase transitions of McKean-Vlasov processes in double-wells landscape, *Stochastics*, 86 (2) (2014), pp. 257-284

♣ Duong, M.H., Tugaut, J., Stationary solutions of the Vlasov-Fokker-Planck equation: Existence, characterization and phase transition, Appl. Math. Lett., 52, 38-45, (2016)

♣ Discrete model, there exist analogues called nonlinear master equations to describe nonlinear pure jump Markov processes. The phase transition is also studied, e.g. the second Schlögl model.[5]

♣ $V \equiv 0$ and $\mathbb{R}^d$ is replaced by the Torus $\mathbb{T}^d$, phase transitions are studied for the McKean-Vlasov equation by using the bifurcation Theory and the Fourier coefficients of the interaction potential F.[6] Their results indicated that non-uniqueness of μ corresponds to the multiplicity of the eigenvalue 0 for the nonlinear elliptic operator

$$\frac{\sigma^2}{2}\Delta\mu - \operatorname{div}(\mu(b)\mu) = 0.$$

[5]♣ Feng, S. and Zheng, X. G., Solutions of a class of non-linear Master equations, *Stoch. Proc. Appl.* **43**,(1992), 65-84. See also Section 15.4 in Chen, M.-F., From Markov Chains to Non-Equilibrium Particle Systems, (2nd Ed.), World Scientific, 2004.

[6]♣ Carrillo, J.A., Gvalani, R.S., Pavliotis, G.A. and A. Schlichting, Long-Time Behaviour and Phase Transitions for the Mckean-Vlasov Equation on the Torus. Arch Rational Mech Anal 235, 635–690 (2020).

For distribution dependent SDE(DDSDE) on $\mathbb{R}^d$

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t \tag{1}$$

♣ Well-posedness, ... of DDSDEs has been intensively studied. [7]

♣ The existence and uniqueness of the invariant probability measures have been investigated.[8]

♣ Stationary distribution (SD): $\mu \in \mathscr{P}$ so that for $\mathscr{L}_{X_0} = \mu$, there is a solution X_t with $\mathscr{L}_{X_t} \equiv \mu$ for all $t \geq 0$.

- ♣ Criteria on the existence of SDs for DDSDEs.
- ♣ How does the phase transition occur ?

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[7] ♣ F.-Y. Wang and his coauthors, X. Zhang and his coauthors, L. Wu and his coauthors, and many other scholars...

[8] ♣ F.-Y. Wang, Exponential Ergodicity for Fully Non-Dissipative McKean-Vlasov SDEs, arXiv: 2101.12562.

♣ W. Liu, L. Wu and C. Zhang, Long-time behaviors of mean-field interaction particle system related to McKean-Vlasov equations,Comm. Math. Phys., 387 (2021), 179-214.

Existence

Freezing $\mathscr{L}_{X_t} = \mu$:

$$\mathrm{d}X_t^\mu = b(X_t^\mu, \mu)\mathrm{d}t + \sigma(X_t^\mu, \mu)\mathrm{d}W_t,$$

If X_t^μ is ergodicity, we have a mapping:

$$\mathcal{T} : \mu \mapsto \mathcal{T}_\mu \text{ (the unique invariant probability measure of } X_t^\mu)$$

♣ The fixed points of $\mathcal{T}$ are SD for X_t.

- ♣ We use Schauder's fixed point theorem:

  A compact mapping in a nonempty, closed and convex subset of a Banach space has a fixed point.

For $r > 0, M > 0$

$$\mathscr{P}^r := \{\mu \in \mathscr{P} \mid \|\mu\|_r := (\mu(|\cdot|^r))^{\frac{1}{r}} < \infty\}$$
$$\mathscr{P}^r_M := \{\mu \in \mathscr{P} \mid \|\mu\|_r \leq M\}.$$

$\|\cdot\|$ the operator norm, $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm.

♣ (H1) There exist constants $r_1 \geq 0, r_2 \geq r_1, r_3 > 0$, $C_1 > 0$, and nonnegative C_2, C_3 such that for any $\mu \in \mathscr{P}^{1+r_2}$

$$2\langle b(x,\mu), x\rangle + (1+r_2-r_1)\|\sigma(x,\mu)\|_{HS}^2 \leq -C_1|x|^{1+r_1} + C_2 + C_3\|\mu\|_{1+r_2}^{r_3}.$$

♣ (H2) For every $n \in \mathbb{N}$ and $\mu \in \mathscr{P}^{1+r_2}$, there is $K_n > 0$ s.t.

$$|b(x,\mu) - b(y,\mu)| + \|\sigma(x,\mu) - \sigma(y,\mu)\|_{HS} \leq K_n|x-y|, \ |x| \vee |y| \leq n.$$

There is a locally bounded function $C_4 : \ [0,+\infty) \to [0,+\infty)$ s.t.

$$|b(x,\mu)| \leq C_4(\|\mu\|_{1+r_2})(1+|x|^{r_1}), \ x \in \mathbb{R}^d, \ \mu \in \mathscr{P}^{1+r_2}.$$

When $r_1 < 1$, we also assume that for any $\mu \in \mathscr{P}^{1+r_2}$

$$\sup_{x\in\mathbb{R}^d} \frac{\|\sigma(x,\mu)\|_{HS}^2}{1+|x|^{2r_1}} < +\infty.$$

♣ (H3) $\forall n \geq 1, M > 0$, and $\mu_m, \mu \in \mathscr{P}_M^{1+r_2}$ with $\mu_m \xrightarrow{w} \mu$, there is

$$\lim_{m\to+\infty} \sup_{|x|\leq n} \left(|b(x,\mu) - b(x,\mu_m)| + \|\sigma(x,\mu) - \sigma(x,\mu_m)\|_{HS}\right) = 0.$$

Existence: regular coefficients

Theorem (Z. 2023 EJP.)

Assume (H1)-(H3) and that σ is non-degenerate on $\mathbb{R}^d \times \mathscr{P}^{1+r_2}$:

$$\sigma(x,\mu)\sigma^*(x,\mu) > 0, \ \ x \in \mathbb{R}^d, \mu \in \mathscr{P}^{1+r_2}.$$

If $r_2 > 0$, $r_3 \leq 1 + r_1$, and $C_1 > C_3$ when $r_3 = 1 + r_1$, then (1) has a stationary distribution.

♣ For $r_3 < 1 + r_1$, we only require that $C_1 > 0$.

♣ Example: Let $d = 1$, $a_1, a_2 \in \mathbb{R}$ with $a_1 a_2 < 0$, $\beta > 0$ and $\alpha > 0$. σ is positive and bounded on $\mathbb{R} \times \mathscr{P}$, locally Lipschitz as in (H2) and satisfies and (H3):

$$\mathrm{d}X_t = -\beta(X_t - a_1)X_t(X_t - a_2)\mathrm{d}t - \alpha \int_{\mathbb{R}} (X_t - y)\mathscr{L}_{X_t}(\mathrm{d}y)\mathrm{d}t$$

$$+ \sigma(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t.$$

Existence: singular coefficients

We consider

$$\mathrm{d}X_t = b_0(X_t, \mathscr{L}_{X_t})\mathrm{d}t + b_1(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t. \tag{2}$$

The drift term b_0 is regular, and b_1 is singular satisfying

♣ (H4) $\exists\ p_1 > d$ so that $\sup_{\mu\in\mathscr{P}^{1+r_2}} \|b_1(\cdot,\mu)\|_{\tilde{L}^p} < \infty$.

For every $n \geq 1$ and $M \geq 1$,

$$\lim_{\nu \xrightarrow{w} \mu \text{ in } \mathscr{P}_M^{1+r_2}} \|(b_1(\cdot,\mu) - b_1(\cdot,\nu))\mathbf{1}_{[|\cdot|\leq n]}\|_{L^p} = 0.$$

♣ (H5) $\forall \mu \in \mathscr{P}^{1+r_2}$, $\sigma(\cdot,\mu)$ is uniformly continuous, and

$\nabla\sigma(\cdot,\mu) \in \tilde{L}^{p_2}$ for some $p_2 > d$, and $\exists \lambda_1, \lambda_2 > 0$ s.t.

$$\lambda_1 \leq (\sigma\sigma^*)(x,\mu) \leq \lambda_2,\ \ x \in \mathbb{R}^d, \mu \in \mathscr{P}^{1+r_2}.$$

$\tilde{L}^p := \left\{ f \in L^p_{loc} \mid \|f\|_{\tilde{L}^p} := \sup_z \|\chi\left(\frac{\cdot - z}{r}\right) f\|_p < \infty \right\}$
with $\chi \in C_c^\infty(\mathbb{R}^d)$ and $\mathbf{1}_{[|x|\leq 1]} \leq \chi \leq \mathbf{1}_{[|x|\leq 2]}$.

Theorem (Z. 2023 EJP.)

Assume that b_0 *satisfies (H1)-(H3) (set* $\sigma \equiv 0$ *there) and satisfies a stronger condition : there are positive constants* C_5, C_6 *such that*

$$|b_0(x,\mu)| \le C_5(1+|x|^{r_1}) + C_6\|\mu\|_{1+r_2}^{\frac{r_3 r_1}{1+r_1}}, \ x \in \mathbb{R}^d, \mu \in \mathscr{P}^{1+r_2},$$

where r_1, r_2, r_3 *are constants from (H1). Assume that* b_1 *satisfies (H4), and* σ *satisfies (H3) (set* $b \equiv 0$ *there) and (H5). If* $r_2 > 0$, $r_3 \le 1 + r_1$, *and* $C_1 > C_3$ *when* $r_3 = 1 + r_1$, *then* (2) *has a stationary distribution.*

We use the Zvonkin transform and local L^p in

♣ Xia, P., Xie, L., Zhang, X. and Zhao, G.: $L^q(L^p)$-theory of stochastic differential equations, *Stoch. Proc. Appl.*, **130**, (2020), 5188–5211.

♣ Xie, L. and Zhang, X.: Ergodicity of stochastic differential equations with jumps and singular coefficients, AIHP, **56**, (2020), 175–229.

Existence: gradient form drift

Consider $\mathcal{T}$ has the following form

$$\mathcal{T}_\mu = \frac{\exp\{-V_0(x) - V(x,\mu)\}}{\int_{\mathbb{R}^d} \exp\{-V_0(x) - V(x,\mu)\}\mathrm{d}x}\mathrm{d}x.$$

Let

$$a : \mathbb{R}^d \times \mathscr{P} \mapsto \mathbb{R}^d \otimes \mathbb{R}^d$$

be a measurable function, and let $a(\cdot,\mu) = (a_{ij}(\cdot,\mu))_{1\le i,j\le d}$ be weakly differentiable. Consider the following differential operator:

$$L_\mu g := \mathrm{div}(a(\mu)\nabla g) - \langle a(\mu)\nabla(V_0 + V(\mu)), \nabla g\rangle,$$

which associates with

$$\begin{aligned}\mathrm{d}X_t = &-a(X_t,\mathscr{L}_{X_t})\nabla(V_0(\cdot) + V(\cdot,\mathscr{L}_{X_t}))(X_t)\mathrm{d}t \\ &+ \mathrm{div}(a(\cdot,\mathscr{L}_{X_t}))(X_t)\mathrm{d}t + \sqrt{2a(X_t,\mathscr{L}_{X_t})}\mathrm{d}W_t\end{aligned}$$

Notation: $\bar{\mu}(\mathrm{d}x) := \frac{e^{-\bar{V}(x)}}{\int_{\mathbb{R}^d} e^{-\bar{V}(x)}\mathrm{d}x}\mathrm{d}x$, $\qquad \|\cdot\|_{W_0}$ weight TV norm.

Assumption (D)

♣ (D1) $e^{-V_0} \in L^1$, $\exists \bar{V}$ so that $e^{-\bar{V}} \in L^1$, and $\exists p > d$ and $q \geq 1$ such that $V_0, \bar{V} \in \mathcal{W}^{1,p}_{q,\bar{\mu}} := \{f \in W^{1,p}_{loc} \mid \nabla f \in L^q(\bar{\mu})\}$.

♣ (D2) $\exists W_0 \geq 1$ such that $W_0 \in L^1(\bar{\mu})$ and

$$V(\cdot, \mu) \in W^{1,p}_{loc},\ \mu \in \mathscr{P}_{W_0}.$$

There exist nonnegative functions F_0, F_1, F_2, F_3 s.t. $F_0 \in L^\infty_{loc}$, $F_2 \in L^q(\bar{\mu}) \bigcap L^p_{loc}$, F_1, F_3 are increasing on $[0, +\infty)$ with $\lim_{r\to 0^+} F_1(r) = 0$, and

$$|V(x,\mu) - V(x,\nu)| \leq F_0(x) F_1(\|\mu - \nu\|_{W_0}),$$

$$|V(x,\bar{\mu})| \leq C(F_0(x) + 1),$$

$$|\nabla V(x,\mu)| \leq F_2(x) F_3(\|\mu\|_{W_0}),\ \mu,\nu \in \mathscr{P}_{W_0}.$$

♣ (D3) There is $F_4 \geq 0$, increasing on $[0, +\infty)$ s.t.

$$-V_0(x) + \beta F_0(x) \leq -\bar{V}(x) + F_4(\beta),\ \beta \geq 0.$$

Assumption (W)

♣ (W1) $\exists\, W \geq 1$ such that $\lim_{|x|\to+\infty} W(x) = +\infty$ and

$$\sup_{x\in\mathbb{R}^d} \frac{W_0(x)}{W(x)} < \infty, \qquad \overline{\lim_{|x|\to+\infty}} \frac{W_0(x)}{W(x)} = 0.$$

♣ (W2) $\exists\, W_1 \in W^{2,1}_{loc}$ and strictly increasing functions G_1, G_2 on $[0,+\infty)$ such that G_2 is convex, and

$$\overline{\lim_{r\to+\infty}} \frac{G_1(r)}{G_2(r)} < 1,$$

$$L_\mu W_1 \leq G_1(\|\mu\|_W) - G_2(W),\ \mu \in \mathscr{P}_W.$$

Theorem (Z. 2023 in progress)

Assume **(H)** *and* $F_0 \in L^1(W_0\bar{\mu})$, $a(\mu) \in \mathcal{W}^{1,p_1}_{\bar{\mu}} \cap L^{q_1}(\bar{\mu})$ *for all* $\mu \in \mathscr{P}_{W_0}$ *and some* $p_1 > d$, $q_1 \in [\frac{q}{q-1}, +\infty]$, *and* **(W)** *holds for* $W_1 \in \mathcal{W}^{2,p_2}_{\bar{\mu}}$ *and* $p_2 \geq \frac{q_1 q}{q_1 q-(q+q_1)} \vee \frac{p_1}{p_1-1}$. *Then* $\mathcal{T}$ *has a fixed point with density in* $\mathcal{W}^{1,p}_{q,\bar{\mu}} \cap L^\infty \cap L^1(W\bar{\mu})$.

Non-uniqueness: locally existence

For $a \in \mathbb{R}^d$, define probability measure μ_a : $\mu_a(f) = \mu(f(\cdot - a))$.
For $\kappa > 0$, and $0 < \gamma < 1 + r_2$,

$$\mathscr{P}^{\gamma}_{a,\kappa} := \{\mu \in \mathscr{P}^{1+r_2} \mid \|\mu_a\|_\gamma \leq \kappa\}.$$

Theorem (Z. 2023 EJP.)

Suppose that the coefficients b, σ satisfy assumptions of the first theorem or $b = b_0 + b_1, \sigma$ satisfy assumptions of the second theorem. Assume that there are $a \in \mathbb{R}^d$, $\gamma \in (0, 1 + r_2)$, $\kappa > 0$ and g on $[0, +\infty)^2$ such that $g(\cdot, w_1)$ is continuous and convex for each $w_1 \geq 0$, and

$$2\langle b(x + a, \mu), x\rangle + \|\sigma(x + a, \mu)\|^2_{HS} \leq -g(|x|^\gamma, \|\mu_a\|_\gamma), \mu \in \mathscr{P}^{1+r_2},$$

$$g(w^\gamma, w_1) > 0, \;\; w \geq \kappa, 0 \leq w_1 \leq \kappa,$$

then (1) has a SD $\mu \in \mathscr{P}^{\gamma}_{a,\kappa}$.

Corollary

If there exist $a_1, a_2 \in \mathbb{R}^d$ *and* $\kappa < \frac{|a_1 - a_2|}{4}$ *such that the above assumptions hold, then (1) has two different stationary probabilities* $\mu_1 \in \mathscr{P}^{\gamma}_{a_1,\kappa}, \mu_2 \in \mathscr{P}^{\gamma}_{a_2,\kappa}$.

♣ Set $b(x,\mu) = -\nabla V(x) - \nabla F * \mu(x)$. We can give a sufficient condition to find SDs around the critical point of V.

♣ **Example:**

Let $d = 1$,

$$\mathrm{d}X_t = -\beta(X_t - a_1)X_t(X_t - a_2)\mathrm{d}t - \alpha \int_{\mathbb{R}} (X_t - y)\mathscr{L}_{X_t}(\mathrm{d}y)\mathrm{d}t + h(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t,$$

α, β, a_1, a_2 satisfy all the conditions of the first example, σ satisfies (H5) and uniformly nondegenerate, and h is a bounded measurable function satisfying (H4). There is $\kappa \in (0, (|a_1| \wedge |a_2|)/2)$ such that

$$\|\sigma\|_\infty^2 + \|h\|_\infty \kappa < 2\beta\kappa^2(\kappa - |a_1 - a_2|)(\kappa - |a_1| \wedge |a_2|).$$

Non-uniqueness: bifurcation

Let $\alpha_\infty \in (0, +\infty)$. For $\alpha \in [0, \alpha_\infty)$, let $0 < \theta \in C^1([0, \alpha_\infty))$, and

$$\tilde{\mathcal{T}}(x; \rho, \alpha) = \frac{\exp\left\{-\theta(\alpha)\, V_0(x) - \alpha\left(V(x, \rho \mathrm{d}x) + \int_{\mathbb{R}^d} K(x, y)\rho(y)\mathrm{d}y\right)\right\}}{\tilde{Z}(\rho; \alpha)}.$$

We first reformulate this problem w.r.t. a reference probability measure. Let $\bar{V}$ be a measurable function with $e^{-\bar{V}} \in L^1$,

$$\bar{\mu} = e^{-\bar{V}}\mathrm{d}x / \int e^{-\bar{V}}\mathrm{d}x.$$

Reformulate $\tilde{\mathcal{T}}$ into the following form

$$\mathcal{T}(x; \rho, \alpha) = \frac{\exp\left\{-\theta(\alpha)\, V_0 - \alpha\left(V(\rho\bar{\mu}) + \int_{\mathbb{R}^d} K(\cdot, y)\rho(y)\bar{\mu}(\mathrm{d}y)\right) + \bar{V}\right\}}{Z(\rho, \alpha)}.$$

♣ give a sufficient condition to determine the changing of the number of fixed points of $\mathcal{T}(\cdot, \alpha)$ as α crossing some $\alpha_0 \in (0, \alpha_\infty)$.

Bifurcation theorem

For a parameter-dependent problems on Banach space $X \times \mathbb{R}$:

$$F(x, \alpha) = 0, \text{ with } (0, \alpha), \alpha \in V \text{ are trivial solutions}$$

where $F \in C(U \times V; X)$ and $0 \in U \subset X$, $V \subset \mathbb{R}$ are open such that $\nabla F(0, \alpha)$ exists and $\nabla F(0, \cdot) \in C(V; \mathcal{L}(X))$.

Definition

Let $\lambda_1, \cdots, \lambda_k$ be all the negative eigenvalues in the 0-group of $\nabla F(0, \alpha)$ with algebraic multiplicities $m_1, \cdots, m_k$, respectively. Denote

$$\sigma_<(\alpha) = (-1)^{\sum_{i=1}^k m_i},$$

and set $\sum_{i=1}^k m_i = 0$ if $k = 0$. If $\nabla F(0, \alpha)$ is an isomorphism on X for $\alpha \in (\alpha_0 - \delta, \alpha_0) \cap (\alpha_0, \alpha_0 + \delta)$ with some $\delta > 0$ and $\sigma_<(\alpha)$ changes at $\alpha = \alpha_0$, then $\nabla F(0, \alpha)$ has an odd crossing number at $\alpha = \alpha_0$.

Theorem (Krasnosel'skii's Bifurcation Theorem)

If 0 is an isolated eigenvalue of finite algebraic multiplicity of $\nabla F(0, \alpha_0)$ and $\nabla F(0, \alpha)$ has an odd crossing number at $\alpha = \alpha_0$, then $(0, \alpha_0)$ is a bifurcation point for $F(x, \alpha) = 0$, i.e. $(0, \alpha_0)$ is a cluster point of nontrivial solutions $(x, \alpha) \in U \times V, x \neq 0$ of $F(x, \alpha) = 0$.

♣ For a fixed point of $\mathcal{T}(\cdot, \alpha)$, saying $\rho_\alpha \bar{\mu}$, let

$$\Phi(\rho, \alpha) = \rho - \mathcal{T}(\rho, \alpha), \qquad \hat{\Phi}(\rho, \alpha) = \rho_\alpha^{-1} \Phi((\rho + 1)\rho_\alpha, \alpha).$$

Then $(0, \alpha)$ is a trivial solution of $\hat{\Phi}$.

♣ We give a bifurcation analysis for $\hat{\Phi} = 0$.

♣ H. Kielhöfer, Bifurcation Theory: An Introduction with Applications to Partial Differential Equations, Second Edition, New-York, Springer, 2014.

♣ (A1) $\sup_{\alpha\in J} e^{-\theta(\alpha)\,V_0} \in L^1$ for any closed, bounded interval J.
$e^{-\bar V} \in L^1$ and $V_1 : \mathbb{R}^d \to \mathbb{R}$, $V_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$V(x,\rho\bar\mu) = V_1(x) + \int_{\mathbb{R}^d} V_2(x,y)\rho(y)\bar\mu(\mathrm{d}y),\ x\in\mathbb{R}^d, \rho\in L^1(\bar\mu),$$

$$\int_{\mathbb{R}^d}\Big(|V_0|^r + |V_1|^r + e^{\beta\| V_2(x,\cdot)\|_{L^2(\bar\mu)}}\Big)\bar\mu(\mathrm{d}x) < +\infty,\ r\ge 1, \beta>0,$$

and $\exists C_0$ on $[0,+\infty)$ is increasing and positive function so that

$$\begin{aligned}-\theta\,V_0(x) &+ \beta_1|V_1(x)| + \beta_2\|V_2(x,\cdot)\|_{L^2(\bar\mu)}\\ &\le -\bar V(x) + C_0(\theta,\beta_1,\beta_2),\ \theta\in R_\theta, \beta_1\in[0,\alpha_\infty), \beta_2\ge 0.\end{aligned}$$

R_θ : the range of $\theta(\cdot)$

$$\pi_\alpha f := f - \mu_\alpha(f),\ f \in L^1(\mu_\alpha), \qquad \mathbf{V}_{2,\alpha} f := \int_{\mathbb{R}^d} V_2(x,y) f(y) \mu_\alpha(\mathrm{d}y).$$

Lemma (local uniqueness)

Assume (A1). If there is $\alpha_0 \in [0, \alpha_\infty)$ *such that* $\mathcal{T}$ *with* $K \equiv 0$ *has a fixed point* $\rho_{\alpha_0} \in L^2(\bar{\mu})$*, and* $I + \alpha_0 \pi_{\alpha_0} \mathbf{V}_{2,\alpha_0} \pi_{\alpha_0}$ *is invertible on* $L^2(\mu_{\alpha_0})$*, then there is* $\delta > 0$ *such that for each* $\alpha \in J_{\alpha_0,\delta}$*, there exists a unique* $\rho_\alpha \in L^2(\bar{\mu})$ *such that* $\Phi(\rho_\alpha, \alpha) = 0$*, and* $J_{\alpha_0,\delta} \ni \alpha \mapsto \rho_\alpha$ *is continuously differentiable in* $L^2(\bar{\mu})$ *with*

$$\sup_{\alpha \in J_{\alpha_0,\delta}} \|\rho_\alpha\|_\infty < +\infty,$$

$$\sup_{\alpha \in J_{\alpha_0,\delta}} |\partial_\alpha \log \rho_\alpha| \in L^r(\bar{\mu}),\ \ r \geq 1,$$

and for any $r \geq 1$*,* $\rho_\alpha, \partial_\alpha \rho_\alpha, \partial_\alpha \log \rho_\alpha$ *are continuous of* α *from* $J_{\alpha_0,\delta}$ *to* $L^r(\bar{\mu})$*.*

♣ (A2) There are measurable functions $K_1 \in L^2(\bar{\mu})$ and $K_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ so that

$$K(x, y) = K_1(y) + K_2(x, y),\ \ x, y \in \mathbb{R}^d,$$

and for any $\beta > 0$,

$$\int_{\mathbb{R}^d} \exp\left\{\beta \|K_2(x, \cdot)\|_{L^2(\bar{\mu})}\right\} \bar{\mu}(\mathrm{d}x) < \infty.$$

♣ (A3) For almost $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} K_2(x, y)\mu_\alpha(\mathrm{d}y) = 0.$$

♣ (A4) There are $\gamma_1, \gamma_2 > 2$ such that $\|V_2\|_{L^2_x L^{\gamma_1}_y}$ and $\|K_2\|_{L^2_x L^{\gamma_2}_y}$ are finite. $(\|V_2\|_{L^2_x L^{\gamma_1}_y} := \int_{\mathbb{R}^d} \|V_2(\cdot, y)\|^{\gamma_1}_{L^2(\bar{\mu})}\bar{\mu}(\mathrm{d}y))$

Let $P(\alpha_0)$ be the eigenprojection of $-\alpha_0\pi_{\alpha_0}(\mathbf{V}_{2,\alpha_0}+\mathbf{K}_{2,\alpha_0})\pi_{\alpha_0}$ associated to the eigenvalue 1:

$$P(\alpha_0)=-\frac{1}{2\pi\mathbf{i}}\int_\Gamma(-\alpha_0\pi_{\alpha_0}(\mathbf{V}_{2,\alpha_0}+\mathbf{K}_{2,\alpha_0})\pi_{\alpha_0}-\eta)^{-1}\mathrm{d}\eta,$$

where $\mathbf{i}=\sqrt{-1}$, and Γ is some simple and closed curve enclosing 1 but no other eigenvalue. Denote

$$\mathcal{H}_0=P(\alpha_0)L^2_{\mathbb{C}}(\mu_{\alpha_0}),\qquad \mathcal{H}_1=(I-P(\alpha_0))L^2_{\mathbb{C}}(\mu_{\alpha_0}).$$

$\mathcal{H}_0$ is finite dimensional. Denote

$$\tilde{A}_0=-P(\alpha_0)\alpha_0\pi_{\alpha_0}(\mathbf{V}_{2,\alpha_0}+\mathbf{K}_{2,\alpha_0})\pi_{\alpha_0}\Big|_{\mathcal{H}_0},\quad \tilde{M}_0=P(\alpha_0)\mathcal{M}_{\partial_\alpha\log\rho_{\alpha_0}}\Big|_{\mathcal{H}_0}.$$

$\tilde{A}_0$ and $\tilde{M}_0$ are matrices on $\mathcal{H}_0$.

Theorem (Z. 2023 in progress)

Assume (A1), (A2) and (A4). Let $\alpha_0 \in [0, \alpha_\infty)$ *such that* $\mathcal{T}$ *with* $K \equiv 0$ *has a fixed point* $\rho_{\alpha_0} \in L^2(\bar{\mu})$ *and* $I + \alpha_0 \pi_{\alpha_0} \mathbf{V}_{2,\alpha_0} \pi_{\alpha_0}$ *is invertible on* $L^2(\mu_{\alpha_0})$*, and let the family* $\{\rho_\alpha\}_{\alpha \in J_{\alpha_0,\delta}}$ *be given by the previous lemma. Suppose that* K_2 *and the probability measure family* $\{\mu_\alpha\}_{\alpha \in J_{\alpha_0,\delta}}$ *satisfy (A3).*

♣ *If* 0 *is an eigenvalue of* $I + \alpha_0 \pi_{\alpha_0}(\mathbf{V}_{2,\alpha_0} + \mathbf{K}_{2,\alpha_0})\pi_{\alpha_0}$*,* $I_{\mathcal{H}_0} + \tilde{M}_0$ *is invertible on* $\mathcal{H}_0$ *and the algebraic multiplicity of the eigenvalue* 0 *of* $(I_{\mathcal{H}_0} + \tilde{M}_0)^{-1}\left(\tilde{A}_0^{-1} - I_{\mathcal{H}_0}\right)$ *is odd, then* $(0, \alpha_0)$ *is a bifurcation point for* $\hat{\Phi} = 0$*, i.e. in any neighbourhood of* $(0, \alpha_0)$*, there is more than one solution of* $\hat{\Phi} = 0$*.*

♣ *In particular, if* 0 *is a semisimple eigenvalue of* $I + \alpha_0 \pi_{\alpha_0}(\mathbf{V}_{2,\alpha_0} + \mathbf{K}_{2,\alpha_0})\pi_{\alpha_0}$ *with odd multiplicity and* $I_{\mathcal{H}_0} + \tilde{M}_0$ *is invertible on* $\mathcal{H}_0$*, then* $(0, \alpha_0)$ *is a bifurcation point for* $\hat{\Phi} = 0$*.*

Back to Dawson's example

$$\tilde{\mathcal{T}}(x;\rho,\alpha) = \frac{\exp\left\{-\frac{2}{\sigma^2}\left(\frac{x^4}{4}-\frac{x^2}{2}\right)-\frac{\beta}{\sigma^2}\int_{\mathbb{R}}(x-y)^2\mu(\mathrm{d}y)\right\}}{\tilde{Z}(\rho,\beta,\sigma)} = \frac{\exp\left\{-\frac{2}{\sigma^2}\left(\frac{x^4}{4}-\frac{x^2}{2}\right)-\frac{\beta}{\sigma^2}x^2+\frac{2\beta}{\sigma^2}x\int_{\mathbb{R}}y\mu(\mathrm{d}y)\right\}}{\tilde{\tilde{Z}}(\rho,\beta,\sigma)}.$$

Here, $V_0(x)=\frac{x^4}{4}-\frac{x^2}{2}$, $V_1(x)=\frac{x^2}{2}$, $V_2\equiv 0$, $K_2(x,y)=xy$, $K_1(y)=\frac{y^2}{2}$, $\alpha=\frac{2\beta}{\sigma^2}$, $\theta(\alpha)=\frac{\alpha}{\beta}$,

$$\rho_\alpha = \frac{1}{\tilde{Z}(\rho,\alpha)}\exp\left\{-\theta(\alpha)\left(\frac{x^4}{4}-\frac{x^2}{2}\right)-\alpha\frac{x^2}{2}\right\}$$

♣ 0 is a semisimple eigenvalue of $I+\alpha_0\pi_{\alpha_0}(\mathbf{V}_{2,\alpha_0}+\mathbf{K}_{2,\alpha_0})\pi_{\alpha_0}$ with odd multiplicity i.f.f. $\alpha_0\int_{\mathbb{R}}x^2\rho_{\alpha_0}(x)\mathrm{d}x=1$. (Dawson's criteria)

♣ $I_{\mathcal{H}_0}+\tilde{M}_0$ is invertible on $\mathcal{H}_0$ i.f.f.

$$0\neq 1+\frac{\alpha_0}{2}\left(-\int x^4\rho_{\alpha_0}+\left(\int x^2\rho_{\alpha_0}\right)^2\right)\left(=\frac{1}{2}+\frac{\alpha_0}{4}+\frac{1}{2\alpha_0}\right).$$

Thank You!